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Diffusion of grains in Coulomb and neutral systems

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Abstract

The problem of normal and anomalous diffusion is formulated on the basis of integral master-type equations with various probability transition functions for diffusion in coordinate space (PTD functions). Grain diffusion in dusty plasmas has a normal character. The dependence of the diffusion coefficient as a function of the grain–grain interaction strength is investigated.

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1. Introduction

Under usual conditions, the stochastic motion of particles leads to a second moment of their space distribution that is linear in time $\langle r^2(t)\rangle \sim Dt$. Such types of diffusion processes play a crucial role in plasmas, including dusty plasmas and neutral systems in various phases. At the same time in many physical, chemical and biological systems deviations from the linear in time dependence of the mean-squared displacement have been experimentally observed. The average square separation of a pair of particles in a turbulent flow grows, according to Richardson's law, with the third power of time [1]. For diffusion, typical for glasses and related complex systems [2], the observed time dependence is slower than linear. These two types of anomalous diffusion obviously are characterized as superdiffusion and subdiffusion. In this paper, we consider the normal diffusion of grains in dusty plasmas and the new approach to anomalous diffusion, which is universal for subdiffusive and superdiffusive systems.

2. Master equation for diffusion

Let us consider diffusion in coordinate space on the basis of the master equation.The structure of this equation is formally similar to the master equation in the momentum space, except for the conservation law in momentum space:

$$\frac{\mathrm{d}f_g(\mathbf{r},t)}{\mathrm{d}t} = \int \mathrm{d}\mathbf{r}'\{W(\mathbf{r},\mathbf{r}')f_g(\mathbf{r}',\mathbf{t}) - W(\mathbf{r}',\mathbf{r})f_g(\mathbf{r},t)\}. \tag{1}$$

Here and below we use for the PTD function the designation W. The probability transition $W(\mathbf{r}, \mathbf{r}')$ describes the probability for a grain to move from the point $\mathbf{r}'$ to the point $\mathbf{r}$ per unit time. We can rewrite this equation in the coordinates $\mathbf{u} = \mathbf{r}' - \mathbf{r}$ and $\mathbf{r}$.

Assuming that the characteristic displacements are small one may expand equation (1) and arrive at the Fokker–Planck form of the equation for the density distribution $f_g(\mathbf{r}, t)$

$$\frac{\mathrm{d}f_g(\mathbf{r}, t)}{\mathrm{d}t} = \frac{\partial}{\partial r_\alpha}\left[A_\alpha(\mathbf{r}) f_g(\mathbf{r}, t) + \frac{\partial}{\partial r_\beta}(D_{\alpha\beta}(\mathbf{r}) f_g(\mathbf{r}, t))\right]. \qquad (2)$$

The coefficients A_α and $D_{\alpha\beta}$, describing the acting force and diffusion, respectively, can be written as functionals of the PTD function W in the form:

$$A_\alpha(\mathbf{r}) = \int \mathrm{d}^s u\, u_\alpha W(\mathbf{u}, \mathbf{r}), \qquad D_{\alpha\beta}(\mathbf{r}) = \frac{1}{2}\int \mathrm{d}^s u\, u_\alpha u_\beta W(\mathbf{u}, \mathbf{r}), \qquad (3)$$

where s is the dimension of coordinate space. For a homogeneous medium, when the r-dependence of the PTD function is absent, the coefficients $A_\alpha = 0$ while the diffusion tensor is diagonal $D_{\alpha\beta} = \delta_{\alpha\beta} D$, where

$$D = \frac{1}{2s}\int \mathrm{d}^s u\, u^2 W(u). \qquad (4)$$

If the structure of the PTD function W leads to divergence of the integral (4) the expansion in equation (1) is not possible. Then we have to apply the methods discussed in [3, 4] and arrive at the cases of anomalous diffusion. In the general case, when the PTD function W depends also on time, the mean-squared displacement $\langle r^2(t)\rangle$ is equal to

$$\langle r^2(t)\rangle = -\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \left\{\frac{\partial^2 f_g(k, z)}{\partial k^2} + \left[\frac{(s-1)}{k} + \delta_{s,1}\delta(k)\right]\frac{\partial f_g(k, z)}{\partial k}\right\}_{|k\to 0} \exp(zt), \qquad (5)$$

where $f_g(k, z)$ is the Fourier–Laplace transform of the solution of the generalized integral diffusion equation with time-dependent $W(\mathbf{r} - \mathbf{r}', t)$.

If the structure of PTD function W provides finiteness of the integral (4), we arrive at the case of normal diffusion. We apply hydrodynamics to consider the normal grain diffusion in dusty plasma with a moderate grain–grain interaction in the next section.

3. Diffusive mode for small perturbations in dusty plasma

Dusty plasmas are composed of electrons, ions, atoms and highly charged grains (or dust particles). The equilibrium charge of grains can be of the order of thousands of electron charges. For rather high grain density, a moderate or strong grain–grain electrostatic interaction can be realized and the mean-squared displacement of dust depends not only on the collision frequency between grains and atoms, but also on the parameter of interaction between the grains and the typical Debye length of the ambient plasma. Numerical calculations of particle dynamics and structure properties for grains in the model of dust particles strongly interacting via the Debye potential have been performed in a number of papers, e.g. [5] (for an overview also see [6, 7]).

In the present paper, we consider the diffusion mode for a weak or moderate interaction between the charged grains in a gaseous plasma with the interaction parameter $\Gamma = Z_{d0}^2 e^2 / bT_d \leqslant 1$, where $Z_{d0}, b = n_d^{-1/3}$ and T_d are the average grain charge number, average distance between the grains and the grain temperature, respectively. For the low-frequency perturbations, the electrons and ions in plasmas can be treated as Boltzmann distributed

$$n_\alpha = n_{\alpha 0}\exp\left(-\frac{q_\alpha \varphi}{T_\alpha}\right), \qquad \alpha = e, i, \qquad (6)$$

where φ, T_α and q_α are the self-consistent potential, the temperatures and charges of the electrons and ions. To describe the dust dynamics we use the continuity equation and the hydrodynamic equation of motion

$$\frac{\partial n_d}{\partial t} + \frac{\partial (n_d \mathbf{v}_d)}{\partial \mathbf{r}} = 0, \qquad \frac{\partial \mathbf{v_d}}{\partial t} + \mathbf{v_d}\frac{\partial \mathbf{v}_d}{\partial \mathbf{r}} + \frac{1}{m_d n_d}\frac{\partial P_d}{\partial \mathbf{r}} + \frac{q_d}{m_d}\frac{\partial \varphi}{\partial \mathbf{r}} + \nu \mathbf{v}_d = 0. \tag{7}$$

Here $\mathbf{v}_d$ is the flow velocity for dust and q_d, m_d and ν are the charge, mass and the collision frequency, respectively, with atoms of the dust particles. The atomic component is considered as an undisturbed medium. The pressure P_d for the weakly interacting dust is determined by the integral over $\mathbf{v}$ with the distribution function of grains:

$$P_d = P_d^K \equiv \frac{m_d}{3}\int \mathrm{d}\mathbf{v}\, v^2 f_d(\mathbf{v}, \mathbf{r}, t). \tag{8}$$

In the general case, the full pressure P_d includes also the term with the pair correlation function responsible for interaction between the grains. The potential φ obeys the Poisson equation:

$$\Delta\varphi = -4\pi\rho, \qquad \rho = (Z_i e n_i - e n_e + q_d n_d). \tag{9}$$

Here e is equal to the modulus of the electron charge. For negatively charged grains $q_d = -Z_d(\mathbf{r})e$. To take into account the dust charge fluctuations, we will use the orbit limited model (OML) and suggest the particles are in local thermal equilibrium

$$\frac{n_e(\mathbf{r}, t)}{n_i(\mathbf{r}, t)} = \alpha\left(1 + \frac{Z_d(\mathbf{r}, t)e^2}{aT_i}\right)\exp\left(\frac{Z_d(\mathbf{r}, t)e^2}{aT_e}\right), \tag{10}$$

where $\alpha \equiv \sqrt{T_\mathrm{i} m_e / T_e m_\mathrm{i}}$ and a is the radius of grains. For simplicity, we neglect from the beginning the grain charge fluctuations. Then from equations (6)–(9) for the small perturbations proportional to $\exp(-\mathrm{i}\omega t + \mathrm{i}\mathbf{kr})$ we obtain for the fluctuation of the grain density n_d^1

$$\left\{-\mathrm{i}\omega\nu + k^2\left(\frac{\omega_{pd}^2}{k^2 + k_D^2} + \frac{1}{m_d}\frac{\partial P_d}{\partial n_d}\right)\right\} n_d^1 = 0. \tag{11}$$

Here k_D is the wave-vector, which provides the screening of the electrostatic field in plasmas by the electrons and ions with the Debye radius $\lambda = 1/k_D$. For $k \ll k_D$, equation (11) describes the specific diffusive mode in dusty plasmas with the diffusion coefficient $\tilde{D}$:

$$\omega = -\mathrm{i}k^2\tilde{D}, \qquad \tilde{D} = D\left[\frac{1}{T_d}\frac{\partial P_d}{\partial n_d} + \frac{\omega_{pd}^2}{k_D^2 v_{Td}^2}\right]. \tag{12}$$

Here $v_{Td} = \sqrt{T_d/m_d}$ and ω_{pd} are the thermal velocity and plasma frequency (for the average charge) of grains and $D = v_{T_d}^2/\nu$ is the diffusion coefficient of the uncharged grains in the ambient atomic media. The diffusion coefficient $\tilde{D}$ can be rewritten as a function of two dimensionless parameters Γ and $\kappa^{-1} = \lambda n_d^{1/3}$

$$\tilde{D} = D\left[1 + \frac{4\pi\Gamma}{\kappa^2}\right], \tag{13}$$

where we suggested that grains are weakly interacting ($\Gamma \leqslant 1$) and $P_d = P_d^K = n_d T_d$. As follows from equation (13) even for $\Gamma \leqslant 1$, the effective diffusion coefficient can be much bigger than D if $\kappa \leqslant 1$. This effect is similar to ambipolar diffusion in electron–ion plasmas. The usual ambipolar mode

$$\omega_a = -\mathrm{i}k^2\frac{(\lambda_e^2 + \lambda_i^2)D_e D_i}{\lambda_e^2 D_e + \lambda_i^2 D_i} \tag{14}$$

exists in dusty plasmas in parallel with the dust diffusion mode (12), but has a bigger decrement than ω. For the case $\lambda_e = \lambda_i$ and $D_e \gg D_i$, equation (14) leads to the standard result $\omega_a = -2\mathrm{i}k^2 D_i$. It can be included in the consideration simultaneously with the dust diffusion mode ω by taking into account the inertial terms in the continuity equations for electrons and ions. The dust diffusive mode can be interpreted as the grain diffusion in the atomic gas with the effective temperature $T_d^* = T_d(1 + 4\pi\Gamma/\kappa^2)$. A tendency to linear increase of the diffusion coefficient as a function of Γ for moderate values of Γ has also been found in [5] by numerical calculations.

Let us take into account the fluctuations of the grain charge by use of equation (10). This model of diffusion in dusty plasmas, taking into account the charging process, leads to renormalization of the diffusion mode and the diffusion coefficient by the simple change $\lambda \to \lambda^*(Z_{d0}) < \lambda$, where $\lambda^*(Z_{d0}) \equiv 1/k_D^*(Z_{d0})$ and $k_D^{*2}(Z_{d0})$ is equal to

$$k_D^{*2}(Z_{d0}) = k_D^2 + \frac{4\pi n_d^0 a T_i}{T^*} L(Z_{d0}, \tau). \qquad (15)$$

In equation (15), $1/T^* \equiv (1/T_e + Z_i/T_i) \simeq Z_i/T_i$, $\tau \equiv T_i/T_e \ll 1$ and the function L equals

$$L(Z_{d0}, \tau) = \frac{\left(1 + \frac{Z_{d0}e^2}{aT_i}\right)}{1 + \tau\left(1 + \frac{Z_{d0}e^2}{aT_i}\right)}. \qquad (16)$$

This renormalization shows that inclusion of charging effects tends to a slower increase of the diffusion coefficient $\tilde{D}$ as a function of Γ than in the model with a fixed grain charge.

Returning to equation (12), we can take into account the influence of interaction between the grains on the diffusive mode via the pressure P_d. For the case $\Gamma \leqslant 1$ and $Z_0^2 n_d^0 \gg n_i^0$ in the Debye–Huckel approximation for P_d, we arrive finally at a modification of equation (13) for the diffusion coefficient

$$\tilde{D} = D\left[1 + \frac{4\pi\Gamma}{\kappa^{*2}} - \frac{\sqrt{\pi}\Gamma^{3/2}}{2}\right], \qquad (17)$$

where $(\kappa^*)^{-1} = \lambda^* n_d^{1/3}$. The derivative $\partial\tilde{D}/\partial\Gamma$ changes sign for $(\kappa^*)^{-2} = \Gamma^{1/2}/8\sqrt{\pi}$. For large values of Γ, the approximation suggested in [8] for $P(\Gamma, \kappa)$ can be used to find $\tilde{D}$. For some values of $\Gamma \gg 1$ and κ the coefficient $\tilde{D}$ is negative, implying an unstable regime of diffusion in the considered model.

It is necessary to underline that the above approach for a weak grain–grain interaction includes both the collective and self-diffusion, which always exist simultaneously in many-particle systems embedded in the ambient media and can be separated only as an approximation (sometimes successfully for the specific models and parameters of plasma).

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